Deforming the Oscillator: Iterative Phases over Parametrizable Closed Paths

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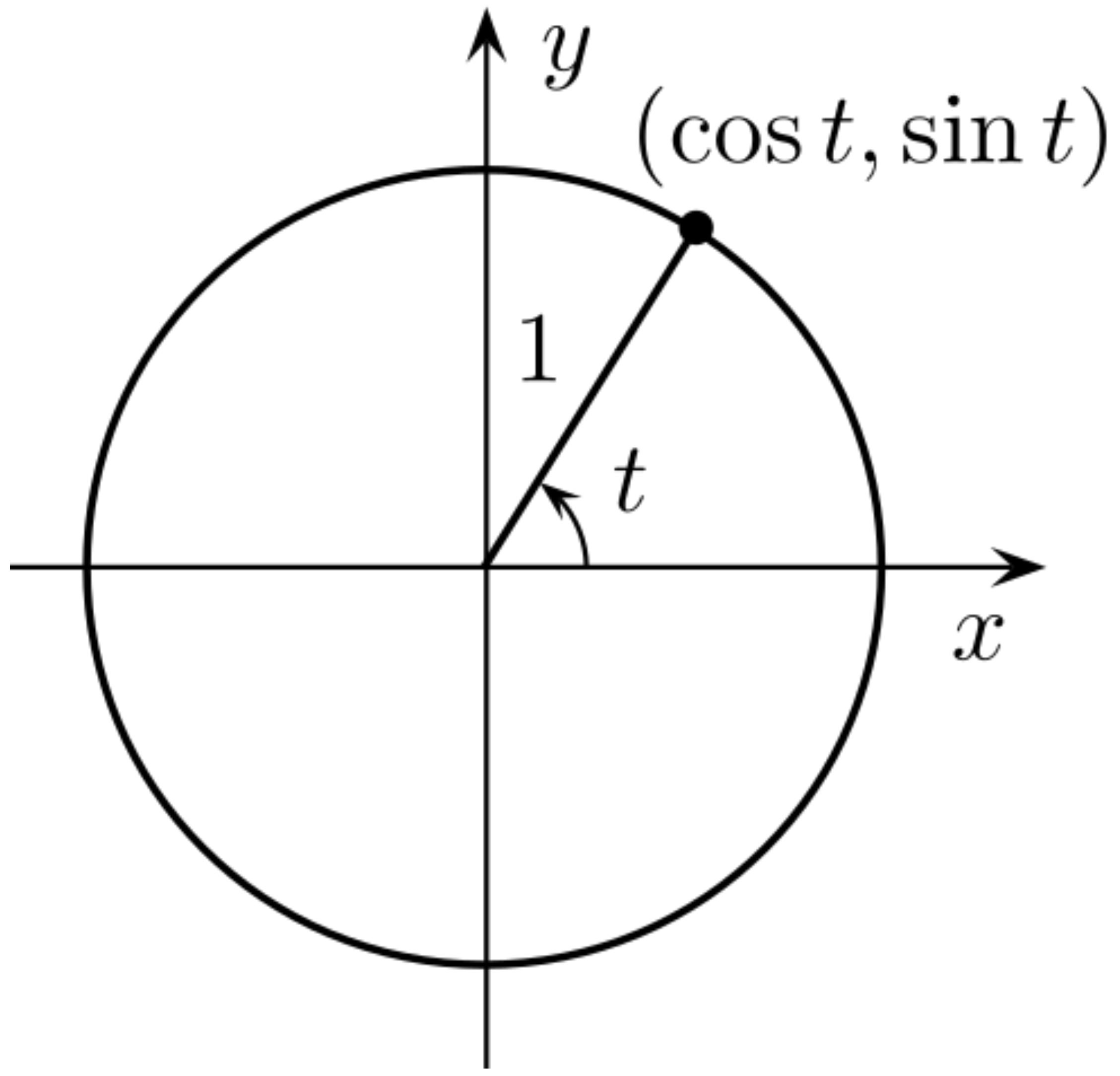


The unit circle can be parameterized by the function

$$o : I \to S^1$$

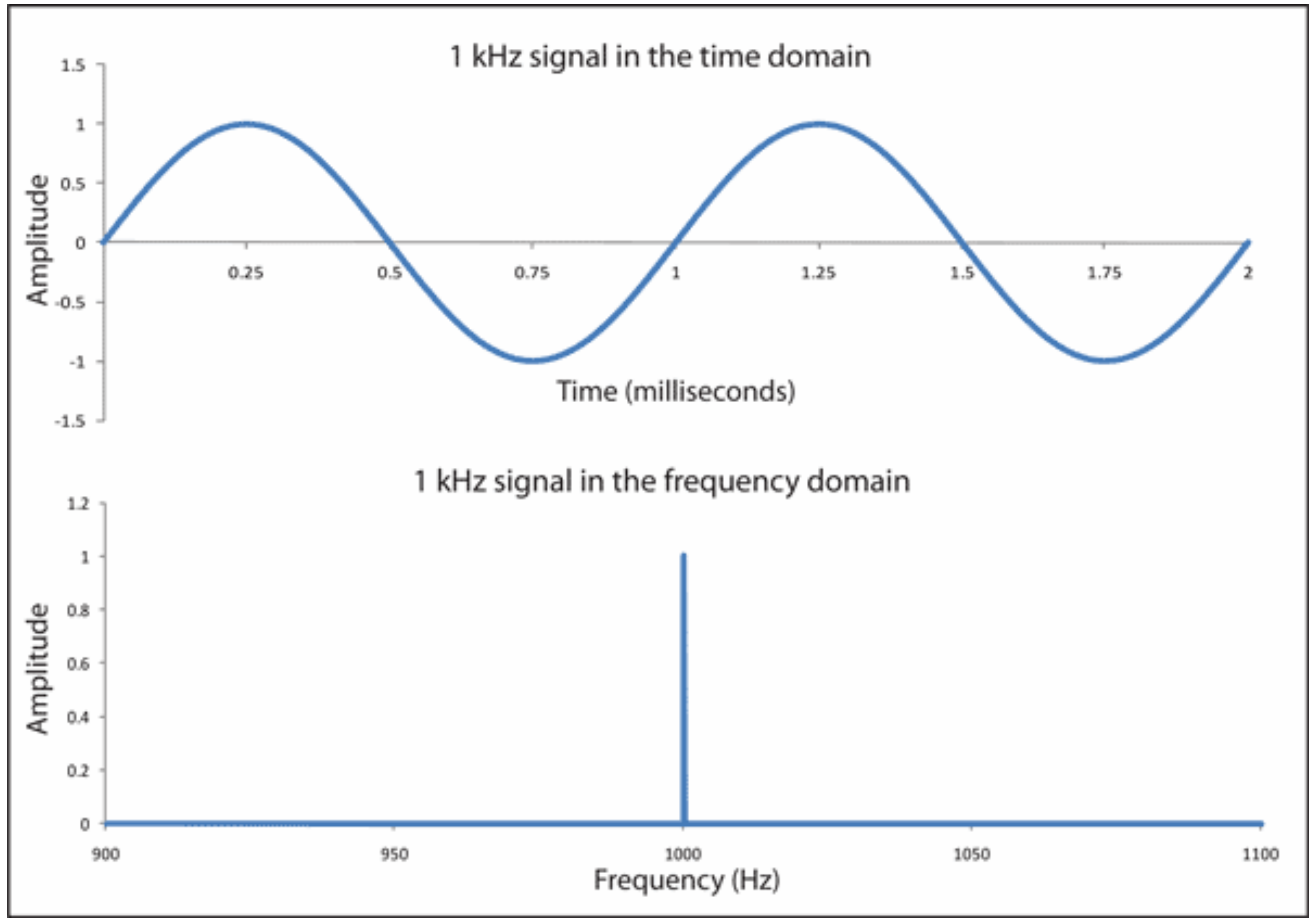
$$o(t) = x + iy = \cos(2\pi t) + i\sin(2\pi t) = e^{2\pi it}$$

with the identification $o(0) = o(1)$, where $I = [0,1] \subset \mathbb{R}$ is the unit interval, and S^1 is the circle (1−sphere). The functions $\mathrm{Re}(o(t)) = x$ and $\mathrm{Im}(o(t)) = y$ are (orthogonal) *projections* of the circle onto the x and y axes.

A *sinusoidal wave* $y : \mathbb{R} \to \mathbb{R}$ is a signal

$$y(t) = \sin(\omega t)$$

where $\omega = 2\pi f$ is the (*angular*) *frequency*. It is a monochromatic (single-frequency) wave.



A (*discrete*) *dynamical system* (X, A) is a space X equipped with a (monoid) action $A = (\mathbb{N}, +)$

$$A \times X \to X$$

which generates the *time evolution* of X. Equivalently, it is an endomorphism $f : X \to X$ with

$$n \cdot x = f^n(x).$$

The set

$$\{f^n(x_0) : n \in \mathbb{N}\}$$

is called the *orbit* of a point $x_0 \in X$.

A function $p : X \to \mathbb{R}$ is called the *output* (or *observation*) *map* of the dynamical system (X, A), and $y = p(x)$ an *output*.

Discretizing the oscillator

Digital audio is a sequence of amplitudes, $y_n \in [-1,1]$[1], quantized to a bit depth, and sampled at a rate f_s.

The discrete oscillator is evaluated at points $t_n = n/f_s$:

$$y_n = \sin(2\pi f \cdot n/f_s) = \sin(2\pi f/f_s \cdot n) = sin(2\pi\Omega \cdot n)$$

where $\Omega = f/f_s$ is the fraction of a cycle progressed per sample.[2] Now, let $x_n = n\Omega$, then

$$x_n = x_{n-1} + \Omega$$

$$y_n = sin(2\pi x_n).$$

Restricting to S^1 gives Essl's eq. (3), $x_n = x_{n-1} + \Omega \mod 1$.[3] As a dynamical system, $X = S^1$ and $f = +\Omega \mod 1$ and $p(\cdot) = sin(2\pi\cdot)$.

[1] We'll focus on discretizing the domain, here; in practice, the codomain [-1,1] is also discretized, using floating point arithmetic.

[2] E.g., with $f = 1kHz$ and $f_s = 44.1kHz$, $\Omega \approx 0.023$.

[3] "Modding out" here is known in audio as *phase wrapping*.

A *sinusoidal oscillator* (S^1, f) is a discrete dynamical system whose evolution $f : S^1 \to S^1$ is

$$x_n = x_{n-1} + \Omega \mod 1$$

with output

$$y_n = \sin(2\pi x_n)$$

A general (not-necessarily sinusoidal) oscillator can be written (Essl, eq. 2)

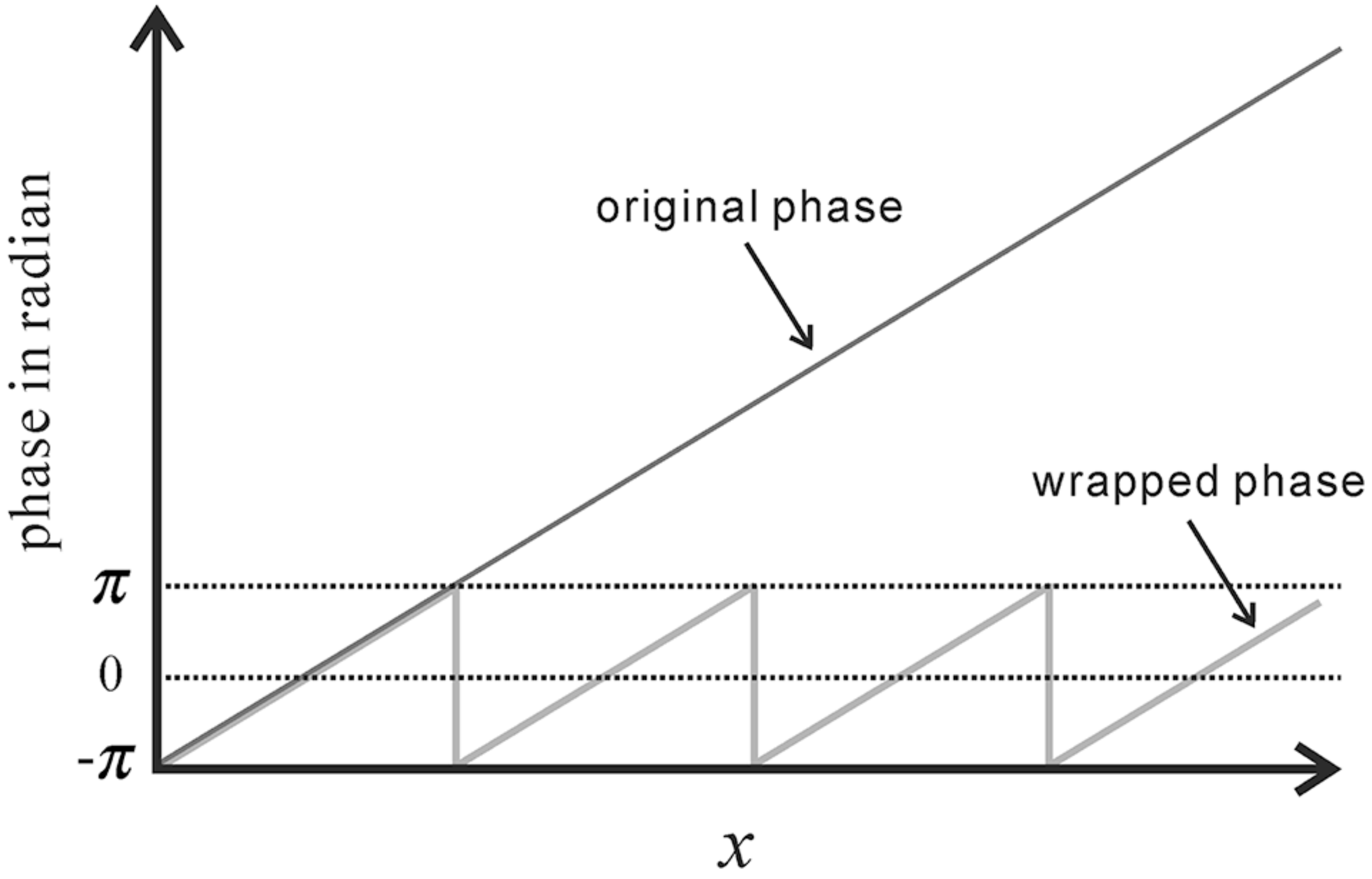
$$\underbrace{y_n}_{\text{Time Series}} = \underbrace{p}_{\text{Projection}} \underbrace{(x_n = f(x_{n-1}) \overbrace{\text{mod } 1}^{\text{Circle Topology}})}_{\text{Iterative Phase Function}}$$

where $f : S^1 \to S^1$ is a circle map. This form decouples the projection and phase update functions.[1]

[1]N.b., if we change the projection function for the oscillator above to the identity, $p = id$, we obtain a *sawtooth wave*. Conversely, if we change the phase update f to $x_n = x_{n-1} + \Omega + (k/2\pi)\sin(2\pi x_{n-1})$, we get *nonlinear feedback frequency modulation.* $k < 1$ locks to rational pitches; $k > 1$ can go chaotic.

Questions: What is the “mod 1” in Essl’s definition of the discrete sinusoidal oscillator doing?

How does it help us generalize to other types of oscillators?



Chp. 6 - Applications in digital holography, Poon and Liu, 2014

Answer: it restricts our phase function to live on the circle[1]:

$$S^1 = [0,1) = \mathbb{R}/\mathbb{Z}$$

Essl's insight: sinusoids already live on the circle, whereas other types of oscillators, which can become unstable, don't necessarily.

Since S^1 is *compact*, restricting to it stabilizes dynamics, in that orbits stay bounded, no matter what f is.

We can also now parameterize our phase function as processes on closed loops.

[1] $\mathbb{R}/\mathbb{Z}$ is read "$\mathbb{R}$ mod $\mathbb{Z}$". "Mod 1" is the quotient map $q : \mathbb{R} \to \mathbb{R}/\mathbb{Z}$ on points, and the universal cover of S^1; on functions it induces *periodization* $h(t) \mapsto$ $(h * Ш)(t)$ $= \Sigma_k h(t-k)$, which is dual to sampling.

This diagram summarizes the setup:

$$\begin{tikzcd}
S^1 \cong \mathbb{R}/\mathbb{Z} \arrow[loop left, "f"] \arrow[r, "g"] \arrow[dr, "p"'] & \mathbb{R}^n \arrow[r, "\pi_2"] \arrow[d, "\pi"] & \mathbb{R}^2 \\
& {[-1,1]} &
\end{tikzcd}$$

$X = S^1$ is the state space; g is a continuous map, the *geometric realization*, into some higher-dimensional ambient space; π_2 is a plotting function; and p and π are projection/output/observation maps, such that the triangle *commutes*:

$$p = \pi \circ g$$

The factorization of p allows one to *deform* (vary) g while keeping $f : S^1 \to S^1$ fixed.

Essl goes on to describe more complex types of oscillators obtained by changing either (or both) of the phase and projection functions:

Oscillator	Phase update $f(x)$ (mod 1)	Projection p	What changed	§
Sine	$x + \Omega$	$\sin(2\pi x)$	–	3.2
Feedback FM (sine circle map)	$x + \Omega + \frac{k}{2\pi}\sin(2\pi x)$	$\sin(2\pi x)$	f	3.3
Deformed carrier	$x + \Omega$	$\pi \circ g$	p	6.1
Feedback FM on a deformed carrier	$x + \Omega + \frac{k}{2\pi}\sin(2\pi x)$	$\pi \circ g$	$f,\ p$	6.2
Deformed modulator	$x + \Omega + H\,p_1(x)$	p_0	the sin inside f	7
Cascade ($i = 0, \ldots, N$)	$x^i + \Omega + H^i\,p_{i+1}\big(f_i(x^i, x^{i+1}, \omega_m^i)\big)$	p_0	iterate	7.1

In this example, f and Ω are fixed and g is deformed (thus p varies); the result is a fixed pitch with changing timbre.

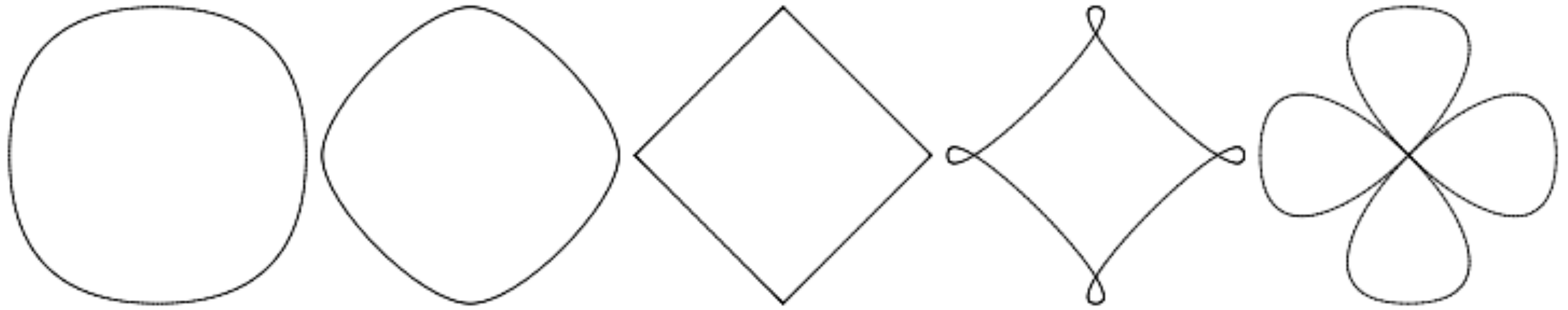


Figure 3: Closed loop tensioned Catmull-Rom splines with four points in a diamond arrangement. $\tau = -1, 0, 1, 2, 5$ from left to right. The center configuration corresponds to a linear connection between control point positions.

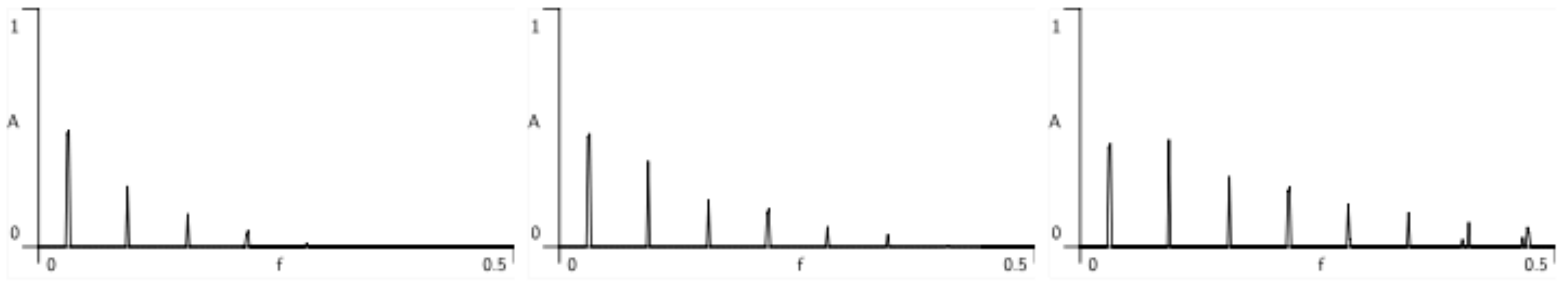


Figure 6: Spectra for the diamond shaped spline for $\tau = -1$, $\tau = 1$ and $\tau = 5$ with $\Omega = 0.03141593$ (corresponding to 1385.4 Hz at 44 100 Hz sampling rate.).

Some potential discussion questions:

- What are some interesting geometric realizations g, other than the spline-based ones that Essl provides?
- (How) can we predict the change in the spectrum based on how g is deformed?
- How might things change if (X, f) were an arbitrary dynamical system, i.e. not an oscillator?

References:

- Georg Essl, *Deforming the Oscillator: Iterative Phases over Parameterizable Closed Paths*, DAFx 2022